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Martin's Axiom and embeddings of upper semi-lattices into the Turing degrees

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ABSTRACT

It is shown that every locally countable upper semi-lattice of cardinality the continuum can be embedded into the Turing degrees, assuming Martin's Axiom.
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1. Introduction

A partial order (upper semi-lattice, lattice) $(L, \leq_L)$ $((L, \leq_L, \vee_L), (L, \leq_L, \vee_L, \wedge_L))$ is *locally countable* if and only if there are at most countably many $\leq_L$-predecessors for each element of L.

The Turing degrees form a locally countable partial order and also an upper semi-lattice (*usl* for short) under the ordering induced by Turing reducibility. Sacks [4] proved that this locally countable partial order is universal under the Continuum Hypothesis (CH), by extending a given countable embedding of an ideal to a larger but still countable one and iterating this procedure for ω_1 many times. This extend-and-iterate strategy also works under Martin's Axiom (MA).

Theorem 1.1 (*[4], Section 3*). *(MA) Every locally countable partial order of cardinality 2^ω can be embedded into the Turing degrees.*

Sacks then conjectured that ZFC suffices.

Conjecture 1.2 (*(C4) in Section 12, [4]*). *ZFC implies Theorem 1.1.*

Although this conjecture remains open, people have found some approximations. One negative approximation by Groszek and Slaman stated that the extend-and-iterate strategy may fail.

Theorem 1.3 (*[2]*). *There is a model M of ZFC $+\ 2^\omega = \omega_2$ in which there is a maximal independent set of Turing degrees of cardinality ω_1.*

Hence in the model above, the independent set cannot be extended to one of cardinality ω_2.

If we replace partial orders in Conjecture 1.2 with usl then we know much more with one exception.

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On the one hand, under CH we have a much stronger positive answer which implies a positive answer even for locally countable lattices.

Theorem 1.4 (*[1]*). *Every locally countable usl of cardinality ω_1 can be embedded into the Turing degrees as an initial segment.*

On the other hand, Groszek and Slaman answer the question under general settings negatively.

Theorem 1.5 (*[2]*). *There exist a locally countable usl $(L, \leq_L, \vee_L)$ of cardinality ω_2 and a model M of $ZFC + 2^\omega = \omega_2$ such that*

$M \models$ *there is no embedding of L into the Turing degrees*.

The exception mentioned is the case under MA.

Conjecture 1.6 (*[2]*). *(MA) Every locally countable upper semi-lattice of cardinality 2^ω can be embedded into the Turing degrees.*

Actually they even asked in [2] whether locally countable upper semi-lattices of cardinality 2^ω can be embedded as initial segments of the Turing degrees, under MA.

But as Shore pointed out to the author, the construction in [1] used perfect forcings which do not satisfy the countable chain condition. Hence Theorem 1.4 cannot be directly generalized. If we try to adapt Sacks' proof of Theorem 1.1, then we may find that Sacks' forcing notion does not preserve the supremum even for countable usl $(L, \leq_L, \vee_L)$ containing a subset $(b_n : n < \omega)$ and an element a such that b_m and b_n are incomparable and $a = b_m \vee_L b_n$ for $m \neq n$. So a different forcing is in need.

In this article, I confirm Conjecture 1.6 by devising a right forcing.

First of all, let us recall some recursion theoretic notions. Fix some recursive bijection from natural numbers onto the collection of finite sets of natural numbers; we write A_n for the image of n under this bijection, i.e. the n-th finite set according to this bijection. A *strong array* (recursive in X) is a sequence of finite sets of natural numbers $(B_n : n \in \omega)$ such that there exists a recursive (an X-recursive) function $h : \omega \to \omega$ with $\forall n(B_n = A_{h(n)} \neq \emptyset)$ and $B_m \cap B_n = \emptyset$ if $m \neq n$. We may identify a strong array (recursive in a given set) with the corresponding function of indices. Given two infinite sets of natural numbers X and Y, Y is *hyperimmune* in X if there is no X-recursive strong array h with $\forall n(A_{h(n)} \cap Y \neq \emptyset)$.

Secondly, recall some set theoretic notions. Given a partial order $(P, \leq_P)$, two elements p and q are *compatible* if there exists r with $r \leq_P p$ and $r \leq_P q$; otherwise they are *incompatible*. A subset $A \subseteq P$ is an *antichain* if its elements are pairwise incompatible. $(P, \leq_P)$ satisfies the *countable chain condition* (c.c.c.) if it has no uncountable antichain. The *order topology* of P is generated by subsets of the form $\{q \in P | q \leq_P p\}$. *Martin's Axiom* states that given a family of dense open sets $(D_\alpha : \alpha < \kappa < 2^\omega)$, there exists a filter G intersecting each D_α.

Thirdly, we identify a finite binary sequence (or string for simplification) ξ with a finite set of natural numbers $X = \{n \in \omega | \xi(n) = 1\}$. So we may write $x \in \xi$, etc. The string extension relation is denoted by $\preccurlyeq$, i.e., $\xi \preccurlyeq \xi'$ if and only if ξ is an initial segment of ξ'.

Finally, for more recursion and set theoretic notions, please refer to [6,3] respectively.

2. Embeddings into the Turing degrees

We introduce a special class of embeddings for the sake of extending a smaller embedding to a larger one. The reason will be made clear by Lemma 2.14 and its proof. We will also supply an informal explanation after proving the extension proposition.

Definition 2.1. Let $(L, \vee_L, \leq_L)$ be a locally countable usl.

(1) An injection $\pi : L \to \mathbb{R}$ is a (Turing) *embedding* if and only if $x \leq_L y \leftrightarrow \pi(x) \leq_T \pi(y)$ and $\pi(x \vee_L y) \equiv_T \pi(x) \oplus \pi(y)$ for $x, y \in L$.
(2) If in addition $\bigcup_{y \in F} \pi(y)$ is hyperimmune in $\pi(x)$ for every finite $F \cup \{x\} \subset L$ with $\forall y \in F(y \not\leq_L x)$, then π is a *mutually hyperimmune* embedding.

Let $\lambda < \kappa < 2^\omega$ be ordinals and $(\kappa, \vee_L, \leq_L)$ be a locally countable usl such that $(\lambda, \vee_L \cap (\lambda \times \lambda \times \lambda), \leq_L \cap (\lambda \times \lambda))$ is an ideal of $(\kappa, \vee_L, \leq_L)$. In addition, without loss of generality assume that 0 is the $<_L$-least element.

Proposition 2.2. *(MA) If $\pi : \lambda \to \mathbb{R}$ is a mutually hyperimmune embedding, then there are $Y_\beta \in \mathbb{R}$ for $\lambda \leq \beta < \kappa$ such that the map*

$$\tilde{\pi} = \pi \cup \{(\beta, Y_\beta) | \lambda \leq \beta < \kappa\}$$

is also a mutually hyperimmune embedding.

Without loss of generality, assume that π is given such that $\lambda > 0$ and $\pi(0) = \omega$. Let $X_\alpha = \pi(\alpha)$ for $\alpha < \lambda$. We shall construct Y_β's for $\beta \in \kappa - \lambda$ as desired. To this end, we shall define a c.c.c. partial order and κ many dense open sets to make the Y_β's satisfy appropriate requirements, like $X_\alpha \leq_T Y_\beta$. Then we shall apply Martin's Axiom to obtain a sufficiently generic filter and extract Y_β's from this filter.

$\tilde{\pi}$ must meet the following requirements:

(1) $\alpha \leq_L \beta \to \tilde{\pi}(\alpha) \leq_T \tilde{\pi}(\beta)$,

(2) $\alpha = \beta \vee_L \gamma \to \tilde{\pi}(\alpha) \leq_T \tilde{\pi}(\beta) \oplus \tilde{\pi}(\gamma)$, and
(3) if $\{\alpha_k | k < n\}$ and β are such that $\forall k < n(\alpha_k \not\leq_L \beta)$ then $\bigcup_{k<n} \tilde{\pi}(\alpha_k)$ is hyperimmune in $\tilde{\pi}(\beta)$.

Note that the mutual hyperimmunity (3) implies the reverse direction of (1), and (2) together with (1) implies $\tilde{\pi}(\beta \vee_L \gamma) \equiv_T \tilde{\pi}(\beta) \oplus \tilde{\pi}(\gamma)$.

First of all, for each $\alpha \in \kappa - \lambda$ fix a function $f_\alpha : \omega \to \kappa$ such that

$$ran(f_\alpha) = \{\beta < \kappa | \beta \neq \alpha \wedge \beta \leq_L \alpha\}.$$

f_α exists as $(\kappa, \vee_L, \leq_L)$ is locally countable and $0 <_L \alpha$.

Let P denote the set of $p = (\varphi_p, I_p, J_p)$ where

(1) φ_p is a finite function, $dom(\varphi_p) \subset \kappa - \lambda$ and $\xi_\alpha^p = \varphi_p(\alpha)$ is a finite binary sequence for each $\alpha \in dom(\varphi_p)$,
(2) I_p and J_p are finite subsets of λ and $\kappa - \lambda$ respectively, and
(3) $J_p = dom(\varphi_p)$ and $\|\varphi_p\| =_{\text{def}} |\varphi_p(\alpha)| = |\varphi_p(\beta)|$ for $\alpha, \beta \in J_p$.

If $p \in P$ and $\alpha \in J_p$ then ξ_α^p is a finite approximation to Y_α. In order to code either X_β (if $\beta < \lambda$) or Y_β (if $\beta \geq \lambda$) where $\beta = f_\alpha(n)$ into Y_α, we define a functional Δ_n such that $\Delta_n(\xi; x) = i$ if and only if

$$\exists s(\langle 0, n, x, s, i\rangle = \min\{\langle 0, n, x, t, j\rangle | \langle 0, n, x, t, j\rangle \in \xi\}).$$

Given $\alpha \geq \lambda$ with $\gamma = \alpha \vee_L \beta, \gamma \geq \lambda$ as λ is an ideal of the usl. To code Y_γ into $Y_\alpha \oplus X_\beta$ (if $\beta < \lambda$) or $Y_\alpha \oplus Y_\beta$ (if $\beta \geq \lambda$), we define another functional Θ such that $\Theta(\xi \oplus \tau; x) = i$ if and only if

$$\exists \sigma(\langle 1, x, i, \sigma\rangle = \min\{\langle 1, x, j, \rho\rangle \in \xi | \rho \preccurlyeq \tau \wedge x < |\rho| \wedge \rho(|\rho| - 1) = 1\}).$$

We call the above bounded variable σ the *witness* of the computation. From now on, we always mean $i < 2$ when we write either $\langle 0, n, x, s, i\rangle$ or $\langle 1, x, i, \sigma\rangle$.

Definition 2.3. For $p, q \in P$, $p \geq_P q$ if and only if

(1) $I_p \subseteq I_q$, $dom(\varphi_p) \subseteq dom(\varphi_q)$ and $\xi_\alpha^p \preccurlyeq \xi_\alpha^q$ for each $\alpha \in dom(\varphi_p)$,
(2) if $\alpha \in dom(\varphi_p)$, $\beta = f_\alpha(n) \in I_p \cup J_p$ and x, i are such that $\Delta_n(\xi_\alpha^p; x) \uparrow$ and $\Delta_n(\xi_\alpha^q; x) \downarrow= i$, then either $\beta \in I_p \wedge X_\beta(x) = i$ or $\beta \in J_p \wedge \xi_\beta^q(x) = i$, and
(3) if $\alpha \in dom(\varphi_p)$, $\{\beta, \gamma\} \subseteq I_p \cup J_p$, $\gamma = \alpha \vee_L \beta$ and x, i are such that either $\beta \in I_p \wedge \Theta(\xi_\alpha^p \oplus X_\beta; x) \uparrow \wedge \Theta(\xi_\alpha^q \oplus X_\beta; x) \downarrow= i$ or $\beta \in J_p \wedge \Theta(\xi_\alpha^p \oplus \xi_\beta^p; x) \uparrow \wedge \Theta(\xi_\alpha^q \oplus \xi_\beta^q; x) \downarrow= i$, then $\xi_\gamma^q(x) = i$.

Intuitively, if $\beta = f_\alpha(n) \in I_p$, then p requires that elements of the form $\langle 0, n, x, s, 1\rangle$ in Y_α code elements of X_β and those of $\langle 0, n, x, s, 0\rangle$ code the complement of X_β. The extra parameter s allows delayed codings and is necessary for preserving mutual hyperimmunity. This mechanism guarantees requirements like $\tilde{\pi}(\beta) \leq_T \tilde{\pi}(\alpha)$, for both $\tilde{\pi}(\beta)$ and its complement will be recursively enumerable in $\tilde{\pi}(\alpha)$.

Given $\gamma = \alpha \vee_L \beta$ and $\lambda \leq \alpha$, we may regard Y_α as a continuous functional (like Turing machines with oracles) and use elements of Y_α of the form $\langle 1, x, i, \sigma\rangle$ with $\sigma \prec X_\beta$ (if $\beta < \lambda$) or $\sigma \prec Y_\beta$ (if $\lambda \leq \beta$) to code $Y_\gamma(x)$. So $\tilde{\pi}(\beta)$ is treated as an oracle for the functional. Definition 2.3 (3) together with the definition of Θ formally describes this mechanism. But Θ ignores computations using a finite oracle like $\tau\hat{\ }\langle 0\rangle$. This is for preserving mutual hyperimmunity (see Lemma 2.14).

Certainly we could tolerate finitely many errors of such codings. Definition 2.3(2–3) impose correctness conditions only on new codings.

I_p and J_p bookkeep indices involved in requirements that p requires stronger conditions to respect.

$\leq_P$ is clearly a reflexive relation. To see that $\leq_P$ defines a partial ordering on P, it suffices to show that $\leq_P$ is transitive.

Lemma 2.4. *$\leq_P$ is transitive.*

Proof. Suppose that $r \leq_P q \leq_P p$. (1) in Definition 2.3 is trivially transitive.

Let $\alpha \in dom(\varphi_p)$, $\beta = f_\alpha(n) \in I_p \cup J_p$ and x, i be such that $\Delta_n(\xi_\alpha^p; x) \uparrow$ and $\Delta_n(\xi_\alpha^r; x) \downarrow= i$. If $\Delta_n(\xi_\alpha^q; x) \downarrow$ then $\Delta_n(\xi_\alpha^q; x) \downarrow= i$, and either $\beta \in I_p \wedge X_\beta(x) = i$ or $\beta \in J_p \wedge \xi_\beta^q(x) = i = \xi_\beta^r(x)$ as $q \leq_P p$. Suppose that $\Delta_n(\xi_\alpha^q; x) \uparrow$. If $\beta \in I_p \subseteq I_q$, then $X_\beta(x) = i$ as $r \leq_P q$. If $\beta \in J_p \subseteq J_q$, then $\xi_\beta^r(x) = \xi_\beta^q(x) = i$ by $r \leq_P q$ again. This shows the transitivity of Definition 2.3(2).

Let $\alpha \in dom(\varphi_p)$, $\{\beta, \gamma\} \subseteq I_p \cup J_p$, $\gamma = \alpha \vee_L \beta$ and x, i be such that either $\beta \in I_p \wedge \Theta(\xi_\alpha^p \oplus X_\beta; x) \uparrow \wedge \Theta(\xi_\alpha^r \oplus X_\beta; x) \downarrow= i$ or $\beta \in J_p \wedge \Theta(\xi_\alpha^p \oplus \xi_\beta^p; x) \uparrow \wedge \Theta(\xi_\alpha^r \oplus \xi_\beta^r; x) \downarrow= i$. Suppose that $\beta \in I_p$. If $\Theta(\xi_\alpha^q \oplus X_\beta; x) \downarrow= i$ then $\xi_\gamma^r(x) = \xi_\gamma^q(x) = i$ as $q \leq_P p$. If $\Theta(\xi_\alpha^q \oplus X_\beta; x) \uparrow$ then $\xi_\gamma^r(x) = i$ as $r \leq_P q$. So in either case $\xi_\gamma^r(x) = i$. Similarly, we also have $\xi_\gamma^r(x) = i$ when $\beta \in J_p$. This shows the transitivity of Definition 2.3(3) and completes the transitivity of $\leq_P$. □

Sometimes we are also interested in fragments of forcing conditions.

Definition 2.5. Let $p \in P$, I and J be finite subsets of λ and $\kappa - \lambda$ respectively. $p \upharpoonright (I, J) = (\varphi_p \upharpoonright J, I_p \cap I, J_p \cap J)$ is called the *restriction* of p to (I, J).

The following observations are trivial but helpful for understanding $\leq_P$.

Lemma 2.6. *If $q \leq_P p$ then $q \upharpoonright (I_p, J_p) \leq_P p$.*

Lemma 2.7. *If $p, q \in P$ are such that $I_p \subseteq I_q, J_p \subseteq J_q$, and*

$$\forall \alpha \in J_p \exists n < \omega(\xi_\alpha^p \hat{\ } \langle 0^n \rangle = \xi_\alpha^q)$$

then $q \leq_P p$.

We verify that P is suitable for MA.

Lemma 2.8. *P satisfies c.c.c.*

Proof. If $(p_\nu : \nu < \omega_1)$ is a sequence from P, then by the Δ-system lemma (see [3, Theorem II.1.6]) there exist a subsequence $(q_\nu : \alpha < \omega_1)$, finite $I \subset \lambda$ and $J \subset \kappa - \lambda$ such that for $\mu < \nu < \omega_1$,

(1) $I = I_{q_\mu} \cap I_{q_\nu}$ and $J = J_{q_\mu} \cap J_{q_\nu}$,
(2) $\xi_\alpha^{q_\mu} = \xi_\alpha^{q_\nu}$ for each $\alpha \in J$, and
(3) $\|\varphi_{q_\mu}\| = \|\varphi_{q_\nu}\|$.

It follows that $(\varphi_{q_\mu} \cup \varphi_{q_\nu}, I_{q_\mu} \cup I_{q_\nu}, I_{q_\mu} \cup I_{q_\nu})$ is a well-defined element of P and extends both q_μ and q_ν for each $\mu, \nu < \omega_1$. Hence P has no uncountable antichains. □

For $\alpha < \kappa$, let

$$D_\alpha = \{p \in P | \alpha \in I_p \cup J_p\}.$$

If $\alpha < \lambda$ then $q = (\varphi_p, I_p \cup \{\alpha\}, J_p) \leq_P p$ for any $p \in P$ and $q \in D_\alpha$. If $\alpha \in \kappa - (\lambda \cup J_p)$ for $p \in P$ then $q = (\varphi_q, I_p, J_p \cup \{\alpha\}) \leq_p p$ where $\varphi_q = \varphi_p \cup \{(\alpha, \langle 0^{\|\varphi_p\|} \rangle)\}$, and $q \in D_\alpha$. So D_α is a dense open subset of P.

If $\lambda \leq \alpha$ then for each $l < \omega$ let

$$D_{\alpha,l} = \{p \in D_\alpha | \text{there are at least } l \text{ many } x \text{ with } \xi_\alpha^p(x) = 1\}.$$

For a given $p \in P$, let $q \leq_P p$ be in D_α. Pick $\varphi : J_q \to 2^{<\omega}$ such that

(1) $\forall \beta, \gamma \in J_q(|\varphi(\beta)| = |\varphi(\gamma)| \wedge \varphi_q(\beta) \preccurlyeq \varphi(\beta))$,
(2) if $\beta \in J_q$ and $x \in \varphi(\beta) - \varphi_q(\beta)$ then $x = \langle 2, y \rangle$ for some y,
(3) there are at least l many x with $\varphi(\alpha)(x) = 1$.

Clearly $r = (\varphi, I_q, J_q) \in D_{\alpha,l}$. As $x \in \varphi(\beta) - \varphi_q(\beta)$ adds neither Δ_n nor Θ computations for $\beta \in J_q$, $r \leq_P q \leq_P p$. Hence $D_{\alpha,l}$ is also dense open.

From the density of $D_{\alpha,l}$ and the assumption that π is mutually hyperimmune, the definition of Θ is reasonable for an oracle Y_α extracted from a filter $G \subset P$ that is sufficiently generic.

By Martin's Axiom, we may fix a filter $G \subset P$ such that

$$\forall \alpha < \kappa \forall l < \omega(G \cap D_\alpha \neq 0 \wedge (\lambda \leq \alpha \to G \cap D_{\alpha,l} \neq 0)).$$

For $\alpha \in \kappa - \lambda$, let

$$Y_\alpha^G = \bigcup \{\xi_\alpha^p | p \in G \wedge \alpha \in dom(\varphi_p)\}.$$

Y_α^G is well-defined as G is a filter.

For $\alpha, \beta < \kappa$ with $\alpha \leq_L \beta$ and $\beta \geq \lambda$, let

$$D_{\alpha,\beta}^0 = D_\alpha \cap D_\beta.$$

$D_{\alpha,\beta}^0$ is dense as so are D_α, D_β. For $n = \min f_\beta^{-1}(\alpha)$ and $x < \omega$, let

$$D_{\alpha,\beta,x}^0 = \{p \in D_{\alpha,\beta}^0 | \Delta_n(\xi_\beta^p; x) \downarrow\}.$$

Given $p \in P$, take $q \in D_{\alpha,\beta}^0$ extending p. By Lemma 2.7, we can safely assume that $x < \|\varphi_q\|$. Let s, i be such that $\langle 0, n, x, s, i \rangle \geq \|\varphi_q\|$ and either $\alpha \in I_q \wedge i = X_\alpha(x)$ or $\alpha \in J_q \wedge \xi_\alpha^q(x) = i$. Pick $\varphi : J_q \to 2^{<\omega}$ such that

(1) $\forall \gamma \in J_q(|\varphi(\gamma)| = \langle 0, n, x, s, i \rangle + 1 \wedge \varphi_q(\gamma) \prec \varphi(\gamma))$,
(2) for $\gamma \in J_q$ and $\|\varphi_q\| \leq y \leq \langle 0, n, x, s, i \rangle$, $\varphi(\gamma)(y) = 1$ if and only if $\gamma = \beta \wedge y = \langle 0, n, x, s, i \rangle$.

Then $(\varphi, I_q, J_q) \leq_P q \leq_P p$. Hence $D_{\alpha,\beta,x}^0$ is dense.

Lemma 2.9. *For $\alpha, \beta < \kappa$ with $\alpha \leq_L \beta$ and $\beta \geq \lambda$, if $G \cap D_{\alpha,\beta,x}^0 \neq 0$ for each $x \in \omega$ then $\alpha < \lambda \to X_\alpha \leq_T Y_\beta^G$ and $\lambda \leq \alpha \to Y_\alpha^G \leq_T Y_\beta^G$.*

Proof. Let $p \in G \cap D_{\alpha,\beta}^0$ and $q_x \in G \cap D_{\alpha,\beta,x}^0$ for each x, and let $n = \min f_\beta^{-1}(\alpha)$. As $\Delta_n(\xi_\beta^{q_x}; x) \downarrow$, $x < \|\varphi_{q_x}\|$. As G is a filter, in G there exists $r_x \leq_P p, q_x$.

Firstly, assume that $\alpha < \lambda$. Let x be such that $\Delta_n(\xi^p_\beta; x)\uparrow$. By the choice of q_x and r_x, $\Delta_n(\xi^{q_x}_\beta; x)\downarrow = \Delta_n(\xi^{r_x}_\beta; x)\downarrow = i$ for some $i < 2$. As $r_x \le_P p$, $i = X_\alpha(x)$ by (2) of Definition 2.3. Hence $X_\alpha(x) = \Delta_n(Y^G_\beta; x)\downarrow$.

As $\Delta_n(\xi^p_\beta; x)\downarrow$ for only finitely many x, $X_\alpha(x) = \Delta_n(Y^G_\beta; x)\downarrow$ for all but finitely many x.

The case where $\lambda \le \alpha$ is similar. □

For $\alpha, \beta, \gamma < \kappa$ with $\alpha \vee_L \beta = \gamma$ and $\alpha, \gamma \ge \lambda$, let

$$D^1_{\alpha,\beta,\gamma} = D_\alpha \cap D_\beta \cap D_\gamma,$$

and for $x < \omega$, let $D^1_{\alpha,\beta,\gamma,x}$ be the set of $p \in D^1_{\alpha,\beta,\gamma}$ such that

$$(\beta < \lambda \to \Theta(\xi^p_\alpha \oplus X_\beta; x)\downarrow) \wedge (\beta \ge \lambda \to \Theta(\xi^p_\alpha \oplus \xi^p_\beta; x)\downarrow).$$

Like $D^0_{\alpha,\beta}$ and $D^0_{\alpha,\beta,x}$, $D^1_{\alpha,\beta,\gamma}$ and $D^1_{\alpha,\beta,\gamma,x}$ are dense open.

Lemma 2.10. *Assume that $G \cap D^1_{\alpha,\beta,\gamma,x} \neq 0$ for each $x \in \omega$. Then $\beta < \lambda \to Y^G_\gamma \le_T Y^G_\alpha \oplus X_\beta$ and $\beta \ge \lambda \to Y^G_\gamma \le_T Y^G_\alpha \oplus Y^G_\beta$.*

Proof. Let $p \in G \cap D^1_{\alpha,\beta,\gamma}$ and $q_x \in G \cap D^1_{\alpha,\beta,\gamma,x}$ for each x. As G is a filter, there exists $r_x \le_P p, q_x$ with $r_x \in G$.

Firstly, assume that $\beta < \lambda$. Let x be such that $\Theta(\xi^p_\alpha \oplus X_\beta; x)\uparrow$. By the definition of $D^1_{\alpha,\beta,\gamma,x}$ and $r_x \le_P q_x$, $\Theta(\xi^{r_x}_\alpha \oplus X_\beta; x)\downarrow = \Theta(\xi^{q_x}_\alpha \oplus X_\beta; x)\downarrow = i$ for some $i < 2$. By (3) of Definition 2.3 and $r_x \le_P p$, $\xi^{r_x}_\gamma(x) = i$. Hence $\Theta(Y^G_\alpha \oplus X_\beta; x)\downarrow = Y^G_\gamma(x)$.

As $\Theta(\xi^p_\alpha \oplus X_\beta; x)\downarrow$ for only finitely many x, $\Theta(Y^G_\alpha \oplus X_\beta; x)\downarrow = Y^G_\gamma(x)$ for all but finitely many x.

The case where $\beta \ge \lambda$ is similar. □

Let I and J be finite subsets of λ and $\kappa - \lambda$ respectively, $\alpha < \kappa$ be such that $\beta \not\le_L \alpha$ for every $\beta \in I \cup J$. We shall define dense open subsets of P which guarantee that $\bigcup_{\beta\in I} X_\beta \cup \bigcup_{\beta\in J} Y^G_\beta$ is hyperimmune in X_α (if $\alpha < \lambda$) or Y^G_α (if $\alpha \ge \lambda$) if G meets these dense sets.

We first study the easier case where $\alpha < \lambda$.

Let $p \in \bigcap_{\beta\in I\cup J} D_\beta$ and $e < \omega$.

Case 1. Either $\Phi_e(X_\alpha)$ is partial, or $\Phi_e(X_\alpha; m)\downarrow = k_m$ with $A_{k_m} = \emptyset$ for some m, or there exist $m < n$ and k_m, k_n such that $\Phi_e(X_\alpha; m) = k_m$, $\Phi_e(X_\alpha; n) = k_n$ and $A_{k_m} \cap A_{k_n} \neq 0$. Define $\bar{p} = p$.

Case 2. Otherwise, let $h = \Phi_e(X_\alpha)$. Then h is total and $(A_{h(n)} : n < \omega)$ is a strong array recursive in X_α.

As π is mutually hyperimmune, there exists n such that $\min A_{h(n)} > \|\varphi_p\|$ and $A_{h(n)} \cap (\bigcup_{\beta\in I} X_\beta) = \emptyset$. Let $m = \max A_{h(n)}$, and let $\bar{p} = (\psi, I_p, J_p)$ where $dom(\psi) = J_p$ and $\psi(\beta) = \xi^p_\beta\hat{}\langle 0^m\rangle$ for each $\beta \in J_p$. Then $\bar{p} \le_P p$.

Lemma 2.11. *If $\bar{p}$ is defined as above, then one of the following holds:*

(1) *$\Phi_e(X_\alpha)$ is partial,*
(2) *$h = \Phi_e(X_\alpha)$ is total but $(A_{h(n)} : n < \omega)$ is not a strong array,*
(3) *$h = \Phi_e(X_\alpha)$ is total and*

$$\exists q \in G(q \le_P \bar{p}) \to \exists n \left(A_{h(n)} \cap \left(\bigcup_{\beta\in I} X_\beta \cup \bigcup_{\beta\in J} Y^G_\beta\right) = \emptyset\right).$$

Proof. Immediate, from the analysis before the lemma. □

For α, I and J as above, define

$$D^2_{\alpha,I,J,e} = \{q \in P \mid \exists p(q \le_P \bar{p})\} \cap \bigcap_{\beta\in I\cup J} D_\beta.$$

$D^2_{\alpha,I,J,e}$ is dense open according to the definition of $\bar{p}$.

Lemma 2.12. *If $G \cap D^2_{\alpha,I,J,e} \neq \emptyset$ for every $e < \omega$, then $\bigcup_{\beta\in I} X_\beta \cup \bigcup_{\beta\in J} Y^G_\beta$ is hyperimmune in X_α.*

Proof. Immediate, from Lemma 2.11. □

Now we assume that $\alpha \ge \lambda$. Let $p \in D_\alpha \cap \bigcap_{\beta\in I\cup J} D_\beta$. We may also assume that $I = \{\beta \in I_p \mid \beta \not\le_L \alpha\}$ and $J = \{\beta \in J_p \mid \beta \not\le_L \alpha\}$.

Let $X = \bigoplus_{\beta\in I_p - I} X_\beta$, and let $Z = \bigcup_{\beta\in I} X_\beta$. As π is mutually hyperimmune and λ is closed under $\vee_L$, Z is hyperimmune in X. Fix $e < \omega$.

Case 1. There exists $q \le_P p$ and $n < \omega$ such that $\Phi_e(\xi^r_\alpha; n)\uparrow$ whenever $r \le_P q$. Then let $\hat{p} = q$.

Case 2. There exists $q \le_P p$ and $m, n, k_m, k_n < \omega$ such that $m \neq n$, $\Phi_e(\xi^q_\alpha; m)\downarrow = k_m$, $\Phi_e(\xi^q_\alpha; n)\downarrow = k_n$ and $A_{k_m} \cap A_{k_n} \neq \emptyset$. Then let $\hat{p} = q$.

Case 3. There exists $q \le_P p$ and $n, k < \omega$ such that $\Phi_e(\xi^q_\alpha; n)\downarrow = k$, $\min A_k > \|\varphi_p\|$ and

$$A_k \cap (Z \cup \{x \mid \exists\beta \in J(\xi^q_\beta(x) = 1)\}) = \emptyset.$$

Then let $\hat{p} = (\varphi, I_q, J_q) \le_P q$ be such that $\max A_k < \|\varphi\|$ and

$$\forall\gamma \in J_q \exists n < \omega(\varphi(\gamma) = \xi^q_\gamma\hat{}\langle 0^n\rangle).$$

We shall argue that the above cases cover all situations.

Lemma 2.13. *If the above cases fail for some $p \in D_\alpha \cap \bigcap_{\beta\in I\cup J} D_\beta$, then there exists a strong array $(B_n : n < \omega)$ recursive in X such that $B_n \cap Z \neq \emptyset$ for each n.*

Proof. We define the B_n's by induction. Let $I^\alpha = I_p - I = \{\beta \in I_p | \beta \leq_L \alpha\}$ and $J^\alpha = J_p - J = \{\beta \in J_p | \beta \leq_L \alpha\}$.

Let B_0 be a nonempty finite subset of Z with $\max B_0 > \|\varphi_p\|$.

If B_n is defined, let $m_n = \max \bigcup_{k\leq n} B_k$, and let l_n be the least l such that

$$\forall\langle 1, x, i, \sigma\rangle(x < |\sigma| < m_n + 2 \to \langle 1, x, i, \sigma\rangle < l_n).$$

Let $\varphi_n(\beta) = \xi^p_\beta{}^\frown\langle 0^{l_n}\rangle$ for all $\beta \in J_p$ and $p_n = (\varphi_n, I_p, J_p)$. Then $p_n \leq_P p$.

Find $q \leq_P p_n \restriction (I^\alpha, J^\alpha)$ such that $I_q = I_p, J_q = J_p$, $\Phi_e(\xi^q_\alpha; u_n) \downarrow = k_n$ for some $u_n, k_n < \omega$, and $\min A_{k_n} > m_n$. q exists because of the failures of Cases 1 and 2, and can be found recursively in X.

Let

$$B_{n+1} = A_{k_n} \cup \{v < \omega | m_n < v < \|\varphi_q\|\}.$$

Then $B_{n+1} \cap (\bigcup_{j\leq n} B_j) = \emptyset$ trivially.

Claim 2.14. *If $A_{k_n} \cap Z = \emptyset$, then there exist $\beta \in J^\alpha$ and $\gamma \in I$ such that $\Theta(\xi^{p_n}_\beta \oplus X_\gamma; x) \uparrow$ and $\Theta(\xi^q_\beta \oplus X_\gamma; x) \downarrow$ for some x, and $\eta = \beta \vee_L \gamma \in J_p$.*

Proof. Assume that the claim does not hold. Define $l = \|\varphi_q\| - \|\varphi_p\|$;

$$\varphi(\beta) = \begin{cases} \xi^q_\beta, & \beta \in J^\alpha; \\ \xi^p_\beta{}^\frown\langle 0^l\rangle, & \beta \in J. \end{cases}$$

Let $\hat{q} = (\varphi, I_p, J_p)$. Below, we show that $\hat{q} \leq_P p_n \leq_P p$ and thus Case 3 holds, contradicting the assumption of the lemma.

Definition 2.3(1) trivially holds for p_n and $\hat{q}$.

For Definition 2.3(2), note that

$$\hat{q} \restriction (I^\alpha, J^\alpha) = q \restriction (I^\alpha, J^\alpha) \leq_P p_n \restriction (I^\alpha, J^\alpha).$$

Moreover, if $\beta \in J$ then $\Delta_n(\varphi(\beta); x) \downarrow = i$ if and only if $\Delta_n(\xi^{p_n}_\beta; x) \downarrow = i$, as $\varphi(\beta) = \xi^p_\beta{}^\frown\langle 0^l\rangle$. It follows that Definition 2.3(2) holds for p_n and $\hat{q}$.

It remains to verify Definition 2.3(3) for p_n and $\hat{q}$.

Firstly, if $\beta \in J$ then $\varphi(\beta)$ codes no computations other than those coded by ξ^p_β, as $\varphi(\beta) = \xi^p_\beta{}^\frown\langle 0^l\rangle$. Hence Definition 2.3(3) holds at β.

Secondly let $\beta \in J^\alpha$ and x be arbitrary.

For $\gamma \in J$, suppose that $\Theta(\xi^{p_n}_\beta \oplus \xi^{p_n}_\gamma; x) \uparrow$ and $\Theta(\varphi(\beta) \oplus \varphi(\gamma); x) \downarrow$. Let σ be the witness of the computation $\Theta(\varphi(\beta) \oplus \varphi(\gamma); x) \downarrow$. By the definition Θ, $\sigma(|\sigma| - 1) = 1$. By $\xi^{p_n}_\beta \preccurlyeq \xi^q_\beta$ and the definition of l_n, $|\sigma| > m_n > \|\varphi_p\|$. But by the definition of $\varphi(\gamma)$, $\varphi(\gamma)(|\sigma| - 1) = 0$ and thus $\sigma \not\preccurlyeq \varphi(\gamma)$, contradicting σ being the witness of the computation. So either $\Theta(\xi^{p_n}_\beta \oplus \xi^{p_n}_\gamma; x) \downarrow$ or $\Theta(\varphi(\beta) \oplus \varphi(\gamma); x) \uparrow$.

For $\gamma \in I$, either $\Theta(\xi^{p_n}_\beta \oplus X_\gamma; x) \uparrow \to \Theta(\varphi(\beta) \oplus X_\gamma; x) \uparrow$ or $\eta = \beta \vee_L \gamma \notin J_p$, as the claim is assumed false.

So Definition 2.3(3) also holds for p_n and $\hat{q}$ as

$$\hat{q} \restriction (I^\alpha, J^\alpha) \leq_P p_n \restriction (I^\alpha, J^\alpha),$$

and we get the desired contradiction. □

If $A_{k_n} \cap Z \neq \emptyset$ then $B_{n+1} \cap Z \neq \emptyset$. Otherwise, let β, γ and x be as in the above claim, and let σ be the witness of the computation $\Theta(\xi^q_\beta \oplus X_\gamma; x)$. By the definition of l_n and that of Θ, $m_n < |\sigma| - 1 \in X_\gamma$. Clearly, $|\sigma| - 1 < \|\varphi_q\|$ and thus $|\sigma| - 1 \in B_{n+1} \cap Z \neq \emptyset$ again.

As B_{n+1} and p_{n+1} are found uniformly X-recursively from B_n and p_n, we get an X-recursive strong array $(B_n : n < \omega)$ as desired. □

As Z is hyperimmune in X, the assumption of the above lemma fails, and Cases 1–3 cover all situations and the set defined below is dense open:

$$D^2_{\alpha,I,J,e} = \{q | \exists p(q \leq_P \hat{p})\} \cap D_\alpha \cap \bigcap_{\beta\in I\cup J} D_\beta.$$

The lemma below summarizes the above argument.

Lemma 2.15. *If $G \cap D^2_{\alpha,I,J,e} \neq \emptyset$ then one of the following holds:*

(1) *$\Phi_e(Y^G_\alpha)$ does not define a strong array,*

(2) $h = \Phi_e(Y_\alpha^G)$ *defines a strong array but*

$$\exists n\left(A_{h(n)} \cap \left(\bigcup_{\beta\in I} X_\beta \cup \bigcup_{\beta\in J} Y_\beta^G\right) = \emptyset\right).$$

So the remaining mutual hyperimmunity is established.

Lemma 2.16. *If $\alpha \geq \lambda$ and $G \cap D^2_{\alpha,I,J,e} \neq \emptyset$ for every $e < \omega$ then $\bigcup_{\beta\in I} X_\beta \cup \bigcup_{\beta\in J} Y_\beta^G$ is hyperimmune in Y_α^G.*

By Martin's Axiom, we may take $G \subset P$, a filter intersecting all dense open sets defined above, and set $Y_\alpha = Y_\alpha^G$ for $\alpha \in \kappa - \lambda$. $\tilde{\pi} = \pi \cup \{(\alpha, Y_\alpha) | \alpha \in \kappa - \lambda\}$ is as desired.

This completes the proof of Proposition 2.2.

Remark 2.17. Lemma 2.14 and its proof explain the necessity of introducing mutually hyperimmune embeddings. Suppose that we did not assume mutual hyperimmunity. We assume that π is merely an embedding and want to extend it. Let $\gamma < \lambda$ and $\alpha, \beta \geq \lambda$ be such that $\alpha \not\leq_L \beta$. So we need to ensure that $\Phi_e(Y_\beta) \neq Y_\alpha$. Assume that we have $p \in P$ with $\alpha, \beta, \gamma \in I_p \cup J_p$. It could be the case that for q with $\xi_\beta^q \succ \xi_\beta^p$ and a new computation $\Phi_e(\xi_\beta^q; x)\downarrow$, ξ_β^q always has some element like $\langle 1, y, i, \sigma\rangle$ where $\sigma \succ X_\gamma$. This coding tuple $\langle 1, y, i, \sigma\rangle$ may always block q from extending p, by having $i \neq \xi_\eta^q(y)$ where $\eta = \beta \vee_L \gamma$. In this way, we may fail to make $\Phi_e(Y_\beta) \neq Y_\alpha$. However, suppose that π is mutually hyperimmune. The definition of Θ requires that σ has some new element from X_γ if the coding is applicable. If we collect such elements, we can have a finite set intersecting X_γ. So we can take advantage of the mutual hyperimmunity.

Theorem 2.18. *(MA) Every locally countable upper semi-lattice of cardinality 2^ω can be embedded into the Turing degrees via a mutually hyperimmune embedding.*

Proof. Let $c = 2^\omega$ and $(c, \vee_L, \leq_L)$ be a locally countable upper semi-lattice. As MA implies that c is regular, there exist ordinals $(\kappa_\alpha : \alpha < c)$ such that

(1) $\sup_{\alpha<c} \kappa_\alpha = c$,
(2) $\kappa_\alpha < \kappa_\beta < c$ if $\alpha < \beta < c$, and
(3) $(\kappa_\alpha, \vee_L \cap \kappa_\alpha^3, \leq_L \cap \kappa_\alpha^2)$ is an ideal of $(c, \vee_L, \leq_L)$ for each $\alpha < c$.

By Proposition 2.2, we can find a family of mutually hyperimmune embeddings $(\pi_\alpha : \alpha < c)$ such that

(1) $dom(\pi_\alpha) = \kappa_\alpha$,
(2) $\pi_\alpha \subset \pi_{\alpha+1}$ and
(3) $\pi_\gamma = \bigcup_{\alpha<\gamma} \pi_\alpha$ if $\gamma < \theta$ is a limit ordinal.

Hence $\pi = \bigcup_{\alpha<c} \pi_\alpha : (c, \vee_L, \leq_L) \to \mathbb{R}$ is a mutually hyperimmune embedding. □

Corollary 2.19. *(MA) $\mathcal{D} = (\mathbf{D}, \vee, \leq)$ is isomorphic to a proper substructure $(\mathbf{S}, \vee, \leq)$, where $\mathbf{D}$ is the universe of Turing degrees.*

Proof. $\mathcal{D} = (\mathbf{D}, \vee, \leq)$ is a locally countable usl of cardinality 2^ω. By the above theorem, there is a mutually hyperimmune embedding $\pi : \mathcal{D} \to \mathcal{D}$. $\mathbf{S} = ran(\pi)$ is as desired, since it contains no non-recursive hyperimmune free degrees. □

3. Remarks

There are several related questions worthy of further study.

On the one hand, one may ask about the situation of embedding lattices. The Abraham–Shore Theorem 1.4 implies a positive answer under CH. But as mentioned, their proof does not work under MA. Furthermore, the typical technique of preserving infima is to make $\pi(a)$ and $\pi(b)$ on $\pi(c)$-pointed trees if $c = a \wedge_L b$. Trees easily break the c.c.c. assumption of MA. So one may need to move to assumptions like PFA.

On the other hand, it is interesting to study finer (e.g. m-, tt-, wtt-type) or coarser (e.g. arithmetic, hyperarithmetic) degree structures. Generic filters produced by MA are generated in a highly non-effective way. So it could be too optimistic to expect parallel positive answers for finer degree structures. For example, the method in this paper cannot be applied to truth table degrees by the following proposition.

Proposition 3.1. *If X is non-recursive and $X \leq_{tt} Y$ then Y is not hyperimmune in X.*

Proof. Let Φ be a truth table functional with $X = \Phi(Y)$. We define an increasing function f recursive in X such that $[f(n), f(n+1) - 1] \cap Y \neq \emptyset$ for each n.

Let $f(0) = 0$. Suppose that $f(n)$ is defined. There are only finitely many binary sequences of length $f(n)$, and for each such σ, $\Phi(\sigma\hat{}\langle 0^\infty\rangle)$ is clearly recursive. Hence there exists m such that

$$\forall\sigma \in 2^{f(n)}(\Phi(\sigma\hat{}\langle 0^m\rangle) \not\prec X).$$

Recursively in X, find the least such m and let $f(n+1) = f(n) + m$. It follows that $[f(n), f(n+1) - 1] \cap Y \neq \emptyset$, as $X = \Phi(Y)$.

So we can define the desired f witnessing the non-X-hyperimmunity of Y. □

But independence results on finer or coarser degree structures may also attract logicians.

Finally if we combine these together, then we will reach an upper bound.

Proposition 3.2. *There is a locally countable lattice of cardinality 2^ω which cannot be embedded into the hyperdegrees.*

Proof. By Shore [5], for each $X \in 2^\omega$ there exists a countable lattice L_X with a least element 0_X and a greatest element 1_X, such that $X \le_h e(0_X)$ for any embedding $e : L_X \to 2^\omega$ with respect to $\le_h$.

Now let L be a locally countable lattice of cardinality 2^ω such that:

(1) L contains a least element 0_L and a non-zero a.
(2) L contains sublattices L_X for each $X \in 2^\omega$.
(3) $x \wedge_L y = 0_L$ if either $x = a$ or $x \in L_X$ and $y \in L_Y$ where $X \neq Y$.
(4) For each finite $S \subset 2^\omega$, L contains two elements 1_S and $1'_S$ such that:
 (a) $1_\emptyset = 0_L$, $1'_\emptyset = a$ and $1_{\{X\}} = 1_X$ where $X \in 2^\omega$,
 (b) $1_{S_0} \wedge 1_{S_1} = 1_{S_0 \cap S_1}$, $1'_{S_0} \wedge 1'_{S_1} = 1'_{S_0 \cap S_1}$ and $1_{S_0} \wedge 1'_{S_1} = 1_{S_0 \cap S_1}$ where $S_0, S_1 \subset 2^\omega$ are both finite,
 (c) if $S = \{X_0, \dots, X_n\}$ and $(x_0, \dots, x_n) \in \prod_{i \le n} L_{X_i}$ then
$$1_S = x_0 \vee_L \cdots \vee_L x_n,\ 1'_S = a \vee_L 1_S.$$
(5) L has no other elements.

If $i : L \to 2^\omega$ is an embedding respecting $\le_h$ then $i(a) \not\le_h i(0_{i(a)})$. But our choice of $L_{i(a)}$ implies that $i(a) \le_h i(0_{i(a)})$.

Hence L cannot be embedded into the hyperdegrees. □

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